

The reversibility and an SPDE for the generalized Fleming-Viot Processes with mutation*

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Abstract

The (Ξ, A) -Fleming-Viot process with mutation is a probability-measure-valued process whose moment dual is similar to that of the classical Fleming-Viot process except that the Kingman's coalescent is replaced by the Ξ -coalescent, the coalescent with simultaneous multiple collisions. We first prove the existence of such a process for general mutation generator A . We then investigate its reversibility. We also study both the weak and strong uniqueness of solution to the associated stochastic partial differential equation.

Keywords: Fleming-Viot process, reversibility, Ξ -coalescent, stochastic partial differential equation, strong uniqueness.

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1 Introduction

The classical Fleming-Viot process is a probability-measure-valued process for mathematical population genetics. It describes the evolution of relative frequencies for different types of alleles in a large population undergoing resampling together with possible mutation, selection and recombination; see Ethier and Kurtz [10] and references therein for earlier work on the classical Fleming-Viot process. When the classical Fleming-Viot process only involves mutation and resampling, it is well-known that its moment dual is a function-valued Markov process governed by the Kingman's coalescent and the mutation semigroup.

During the past ten years, more general coalescents have been proposed and studied by many authors. For examples, the Λ -coalescent (cf. Pitman [20] and Sagitov [21]) is a coalescent with possible multiple collisions and the Ξ -coalescent (cf. Möhle and Sagitov [19] and Schweinsberg

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[23]) is a coalescent with possible simultaneous multiple collisions. It is then interesting to know whether there exists a generalized Fleming-Viot type probability-measure-valued process whose dual is a function-valued process evolving in the same way as the classical Fleming-Viot dual but with the Kingman's coalescent replaced by the Ξ -coalescent.

Such a generalized Fleming-Viot process was first considered by Donnelly and Kurtz [9] and Hiraba [13]. When the spatial motion of the particle is negated, namely, the mutation is 0, it has also been studied by Bertoin and Le Gall ([2], [3], [4], [5]) and Birkner et al [7]. In particular, a special form of such process is constructed in [4] using the weak solution flow of a stochastic equation driven by a Poisson random measure. Generalized Fleming-Viot processes with parent independent jump mutation operators are constructed by Dawson and Li [8] as strong solutions of stochastic equations driven by time-space white noises and Poisson random measures. The classical Fleming-Viot process with Laplacian mutation operator is characterized by Xiong [25] as the strong solution of an SPDE driven by a time-space white noise. The common feature of the approaches of [4], [8] and [25] is to consider the processes of distributions of the measure-valued processes instead of their density processes. In fact, the processes studied in [4] and [8] are usually not absolutely continuous. Similar stochastic equations for Dawson-Watanabe superprocesses have also been studied in [4], [5], [8] and [25].

This problem is also studied in the recent work of Birkner et al [6]. When the mutation generator A is the generator for a pure jump Markov process, two constructions of the (Ξ, A) -Fleming-Viot process are found in [6]. One construction is based on modification of the lookdown scheme of [9] applied to exchangeable particle systems for the classical Fleming-Viot process. The (Ξ, A) -Fleming-Viot process arises as the pathwise almost sure limit of the empirical measure for the exchangeable particle system. The other construction is based on the Hille-Yosida theorem. The resulted process gives an example of probability-measure-valued superprocess of jump diffusion type.

In this paper, we further study the existence and various properties of this generalized Fleming-Viot process. We first formulate a well-posed martingale problem and show that the (Ξ, A) -Fleming-Viot process X with general mutation generator is the unique solution to such a martingale problem. We then show that X has a unique invariant measure if the mutation process allows a unique invariant measure.

The reversibility of a population genetic model is an important issue for statistical inference. The reversibility for the classical Fleming-Viot process has been investigated in Li et al [18] using Dirichlet forms and in Handa [12] and Schmuland and Sun [24] via cocycle identity. The reversibility for an interacting classical Fleming-Viot process is studied in Feng et al [11]. We also consider the reversibility for (Ξ, A) -Fleming-Viot process in this paper. By adapting the approach of [18] we first show that for the (Ξ, A) -Fleming-Viot process X to be reversible, the mutation generator A is necessary a parent independent jump generator. Furthermore, if the type space contains at least three points or it contains two points with non identical mutation rates to them, to be reversible the Ξ -coalescent for the Fleming-Viot process has to degenerate into the Kingman's coalescent. When the type space contains exactly two points with equal mutation rates to them, we show that the above-mentioned result is still valid for several examples where explicit computations can be carried out.

When the mutation generator A is the one-dimensional Laplacian operator, we further study the SPDEs associated with the (Ξ, A) -Fleming-Viot process. In order to establish the strong uniqueness we associate the SPDE to a backward SDE and then prove the pathwise uniqueness of the backward SDE using a Yamada-Watanabe type argument. Such an approach was first proposed in [25] to prove the strong uniqueness of SPDE arising from super-Brownian motion

and Fleming-Viot process over the real line.

The rest of this article is organized as follows. In Section 2 we first introduce the (Ξ, A) -coalescent, which serves as the dual to the (Ξ, A) -Fleming-Viot process. In Section 3, we give a new construction of the (Ξ, A) -Fleming-Viot with general mutation generator A . In Section 4, we study the ergodicity and the reversibility of this process. Finally, in Section 5 we study an SPDE associated to the (Ξ, A) -Fleming-Viot process with A being the Laplacian operator. We prove the strong uniqueness of the solution to this nonlinear SPDE driven by a Brownian sheet and a Poisson random measure.

2 The (Ξ, A) -coalescent

We first borrow some notation from Bertoin [1]. Put $[n] := \{1, \dots, n\}$ and $[\infty] := \{1, 2, \dots\}$. A *partition* of $D \subset [\infty]$ is a countable collection $\pi = \{\pi_i, i = 1, 2, \dots\}$ of disjoint *blocks* such that $\cup_i \pi_i = D$ and $\min \pi_i < \min \pi_j$ for $i < j$. Let \mathcal{P}_n denote the set of partitions of $[n]$ and \mathcal{P}_∞ denote the set of partitions of $[\infty]$. Write $\mathbf{0}_{[n]} := \{\{1\}, \dots, \{n\}\}$ for the partition of $[n]$ consisting of singletons.

Given a partition $\pi \in \mathcal{P}_n$ for some n and $\pi' \in \mathcal{P}_k$ with $|\pi| \leq k$ where $|\pi|$ denotes the cardinality of π , the *coagulation* of π by π' , denoted by $\text{Coag}(\pi, \pi')$, is defined as the following partition of $[n]$,

$$\pi'' = \left\{ \pi_j'' := \cup_{i \in \pi_j'} \pi_i : j = 1, \dots, |\pi'| \right\}.$$

For example, for $\pi = \{\{1, 3\}, \{2\}, \{4, 5, 9\}, \{6, 8\}, \{7\}\}$ and $\pi' = \{\{1, 5, 6\}, \{2, 3, 4\}\}$, we have

$$\text{Coag}(\pi, \pi') = \{\{1, 3, 7\}, \{2, 4, 5, 6, 8, 9\}\}.$$

Given a partition π with $|\pi| = b$ and a sequence of positive integers s, k_1, \dots, k_r such that $k_i \geq 2, i = 1, \dots, r$ and $b = s + \sum_{i=1}^r k_i$, we say a partition π'' is obtained by a $(b; k_1, \dots, k_r, s)$ -collision of π if $\pi'' = \text{Coag}(\pi, \pi')$ for some partition π' such that

$$\{|\pi'_i| : i = 1, \dots, |\pi'|\} = \{k_1, \dots, k_r, k_{r+1}, \dots, k_{r+s}\},$$

where $k_{r+1} = \dots = k_{r+s} = 1$, i.e. π'' is a merger of the b blocks of π into $r + s$ blocks in which s blocks remain unchanged and the other r blocks contain k_1, \dots, k_r blocks from π .

The Ξ -coalescent is a \mathcal{P}_∞ -valued coalescent $\Pi_\infty = (\Pi_\infty(t))_{t \geq 0}$ starting from partition $\Pi_\infty(0) \in \mathcal{P}_\infty$ such that for any $n \in [\infty]$, its restriction to $[n]$, $\Pi_n = (\Pi_n(t))_{t \geq 0}$ is a Markov chain and that given $\Pi_n(t)$ has b -blocks, each $(b; k_1, \dots, k_r, s)$ -collision occurs at rate $\lambda_{b; k_1, \dots, k_r, s}$. For the Ξ -coalescent to be well defined, it is sufficient and necessary that there is a nonzero finite measure $\Xi = \Xi_0 + \sigma^2 \delta_{\mathbf{0}}$ on the infinite simplex

$$\Delta = \left\{ \mathbf{x} = (x_1, x_2, \dots) : x_1 \geq x_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} x_i \leq 1 \right\}$$

such that Ξ_0 has no atom at $\mathbf{0}$, $\delta_{\mathbf{0}}$ denotes a point mass at $\mathbf{0}$, $\sigma \geq 0$ is a constant and

$$\lambda_{b; k_1, \dots, k_r, s} = \sigma^2 1_{\{r=1, k_1=2\}} + \beta_{b; k_1, \dots, k_r, s},$$

where

$$\beta_{b; k_1, \dots, k_r, s} = \int_{\Delta} \sum_{l=0}^s \sum_{i_1 \neq \dots \neq i_{r+l}} \binom{s}{l} x_{i_1}^{k_1} \dots x_{i_r}^{k_r} x_{i_{r+1}} \dots x_{i_{r+l}} \left(1 - \sum_{j=1}^{\infty} x_j \right)^{s-l} \frac{\Xi_0(d\mathbf{x})}{\sum_{j=1}^{\infty} x_j^2} \quad (2.1)$$

denotes the rate of simultaneous multiple coalescent with Ξ_0 being the measure of multiple coagulation and σ^2 denotes the rate of binary coagulation. As a result, the coagulation rates satisfy the *consistency condition*

$$\lambda_{b;k_1,\dots,k_r;s} = \sum_{m=1}^r \lambda_{b+1;k_1,\dots,k_{m-1},k_m+1,k_{m+1},\dots,k_r;s} + s\lambda_{b+1;k_1,\dots,k_r,2;s-1} + \lambda_{b+1;k_1,\dots,k_r;s+1}.$$

See Schweinsberg [23].

When the measure Ξ is supported on $[0, 1]$, the corresponding coalescent involves at most multiple collisions. Such Ξ -coalescents are also called Λ -coalescent.

A Poisson process construction of Ξ -coalescent is given in [23] as follows. For each $\mathbf{x} = (x_1, x_2, \dots) \in \Delta$ let $P_{\mathbf{x}}$ be a probability measure on \mathbb{Z}^∞ such that

$$P_{\mathbf{x}}\{\mathbf{z} = (z_1, \dots) \in \mathbb{Z}^\infty : z_i = j\} = x_j,$$

$$P_{\mathbf{x}}\{\mathbf{z} : z_i = -i\} = 1 - \sum_{j=1}^{\infty} x_j$$

and for any $n \in [\infty]$ and $(k_1, \dots, k_n) \in \mathbb{Z}^n$,

$$P_{\mathbf{x}}\{\mathbf{z} : z_i = k_i, i = 1, \dots, n\} = \prod_{i=1}^n P_{\mathbf{x}}\{\mathbf{z} : z_i = k_i\}.$$

Then define a measure L on \mathbb{Z}^∞ by

$$L(B) := \int_{\Delta} P_{\mathbf{x}}(B) \frac{\Xi_0(d\mathbf{x})}{\sum_{j=1}^{\infty} x_j^2} + \sigma^2 \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} 1_{\mathbf{z}_{ij} \in B}$$

where $\mathbf{z}_{ij} = (z_1, \dots)$ with $z_i = z_j = 1$ and $z_k = -k$ for $k \notin \{i, j\}$. Let $(e(t))_{t \geq 0}$ be a \mathbb{Z}^∞ -valued Poisson point process with characteristic measure L . Let $(e_n(t))_{t \geq 0}$ be the process $(e(t))_{t \geq 0}$ restricted to \mathbb{Z}^n . Notice that e and e_n can be identified as \mathcal{P}_∞ -valued and \mathcal{P}_n -valued process, respectively, in the obvious way. Then the Ξ -coalescent Π_n can be constructed using e_n recursively. Given $\Pi_n(0) \in \mathcal{P}_n$ and suppose that $\Pi_n(s)$ has been constructed for $0 \leq s \leq t$. Let $T > t$ be the first jumping time for e_n after t . Then define $\Pi_n(s) = \Pi_n(t)$ for $t < s < T$ and $\Pi_n(T) = \text{Coag}(\Pi_n(t), e_n(T))$. See [23] and references therein for more detailed inductions on the Ξ -coalescent.

Let E be a Polish space containing at least two points. Let $B(E)$ be the set of bounded functions on E . Let $M_1(E)$ be the space of Borel probability measures on E . Then $M_1(E)$ endowed with the topology of weak convergence is a Polish space. For $\mu \in M_1(E)$ and $f \in B(E)$, write $\langle \mu, f \rangle = \mu(f) := \int f d\mu$. For $n \geq 1$ and $f \in B(E^n)$, let

$$G_{n,f}(\mu) = G_\mu(n, f) = \langle \mu^n, f \rangle. \quad (2.2)$$

Let Z be a Markov process with state space E , Feller transition semigroup (P_t) and generator $(A, \mathcal{D}(A))$. Process Z describes the mutation mechanism for the Fleming-Viot process.

Given a function g on E^n , for each partition $\pi = \{\pi_i, i = 1, \dots, |\pi|\}$ on $[n]$, we define $\Phi_\pi g$ as a function on $E^{|\pi|}$ such that $\Phi_\pi g(x_1, \dots, x_{|\pi|}) = g(x_{i_1}, \dots, x_{i_n})$ with $i_j = k$ for $i_j \in \pi_k, j \in [n]$. For example, if g is a function on E^6 and $\pi = \{\{1, 4\}, \{2, 3, 6\}, \{5\}\}$, then

$$\Phi_\pi g(x_1, x_2, x_3) = g(x_1, x_2, x_2, x_1, x_3, x_2).$$

For $1 \leq i < j \leq n$ we write Φ_{ij} for Φ_π with

$$\pi = \{\{1\}, \dots, \{i-1\}, \{i, j\}, \{i+1\}, \dots, \{j-1\}, \{j+1\}, \dots, \{n\}\}.$$

The (Ξ, A) -coalescent $(M, Z) = (M_t, Z_t)_{t \geq 0}$ with initial value $(M_0, Z_0) = (n, f)$, $f \in B(E^n)$ is a $\cup_{k=1}^n \{k\} \times B(E^k)$ -valued Markov process defined as follows. Let Π_n with $\Pi_n(0) = \mathbf{0}_{[n]}$ be the Ξ -coalescent defined before. For any $f \in C(E^n)$, the set of bounded continuous functions on E^n equipped with the supremum norm, define semigroup $(P_t^{(n)})$ by

$$P_t^{(n)} f(x_1, \dots, x_n) := \int_{E^n} f(\xi_1, \dots, \xi_n) \prod_{i=1}^n P_t(x_i, d\xi_i).$$

Then $M := |\Pi_n|$ and Z can be defined iteratively as follows. Write $T_0 = 0$ and $T_1 < T_2 < \dots$ for the sequence of ordered jumping times for Π_n . For any $T_i < t < T_{i+1}$, define

$$Z_t := P_{t-T_i}^{(M_{T_i})} Z_{T_i} \text{ and } Z_{T_{i+1}} = \Phi_{\pi'} Z_{T_{i+1}-} \text{ if } \Pi_n(T_{i+1}) = \text{Coag}(\Pi_n(T_{i+1}-), \pi').$$

For any $\pi \in \mathcal{P}_n$, $\pi \neq \mathbf{0}_{[n]}$, write $\beta_\pi := \beta_{n; k_1, \dots, k_r; s}$ where $(n; k_1, \dots, k_r; s)$ is uniquely determined by

$$\{k_1, \dots, k_r, 1, \dots, 1\} = \{|\pi_i| : 1 \leq i \leq |\pi|\}.$$

For fixed $\mu \in M_1(E)$ and the function G_μ given by (2.2) we define for any $f \in \mathcal{D}(A^{(n)})$,

$$\begin{aligned} L^* G_\mu(n, f) = & G_\mu(n, A^{(n)} f) + \sigma^2 \sum_{i < j} \left[G_\mu(n-1, \Phi_{ij} f) - G_\mu(n, f) \right] \\ & + \sum_{\pi \in \mathcal{P}_n \setminus \{\mathbf{0}_{[n]}\}} \beta_\pi \left[G_\mu(|\pi|, \Phi_\pi f) - G_\mu(n, f) \right], \end{aligned}$$

where $A^{(n)}$ denotes the generator of $(P_t^{(n)})$. Clearly, L^* is the generator for the Markov process (M_t, Z_t) .

3 Existence of the (Ξ, A) -Fleming-Viot process

We call an $M_1(E)$ -valued Markov process X a (Ξ, A) -Fleming-Viot process with Ξ -resampling mechanism and mutation generator A if for any $n \in \mathbb{N}$, $f \in B(E^n)$, its moment is determined by

$$\mathbb{E} G_{n,f}(X_t) = \mathbb{E} G_{X_0}(M_t, Z_t) \quad (3.1)$$

where (M, Z) denotes the (Ξ, A) -coalescent with initial value (n, f) , which is defined in the previous section. Since process X is probability-measure-valued, its distribution is uniquely determined by (3.1).

The (Ξ, A) -Fleming-Viot process is constructed in Birkner et al [6] for generator A of jump type i.e.

$$A f(x) = r \int_E (f(y) - f(x)) q(x, dy),$$

where f is a bounded function on E , $q(x, dy)$ is a Feller transition function and $r > 0$ is the global mutation rate.

In this section, we want to show that the desired probability-measure-valued process is well defined for any Feller generator A with transition semigroup (P_t) on $C(E)$. To this end, we want to show that the (Ξ, A) -Fleming-Viot process is the unique solution to a martingale problem. Let $D_{M_1}[0, \infty)$ be the space of càdlàg paths from $[0, \infty)$ to $M_1(E)$ furnished with the Skorohod topology.

Let $\mathcal{D}_1 \subset C(M_1(E))$ be the linear span of the functions of form (2.2) with $n \geq 1$ and $f \in \mathcal{D}(A^{(n)})$. Let L_0 be the linear operator from \mathcal{D}_1 to $C(M_1(E))$ defined by

$$L_0 G_{n,f}(\mu) = \langle \mu^n, A^{(n)} f \rangle + \sigma^2 \sum_{1 \leq i < j \leq n} \left[\langle \mu^{n-1}, \Phi_{ij} f \rangle - \langle \mu^n, f \rangle \right] \quad (3.2)$$

for $\mu \in M_1(E)$.

Let $C_{M_1}[0, \infty)$ be the space of continuous paths from $[0, \infty)$ to $M_1(E)$ furnished with the topology of uniform convergence. The next theorem follows from Theorem 3.2 of Ethier and Kurtz [10].

Theorem 3.1 *The (L_0, \mathcal{D}_1) -martingale problem in $C_{M_1}[0, \infty)$ is well-posed.*

Note that the solution to the (L_0, \mathcal{D}_1) -martingale problem is the well-known classical Fleming-Viot process. In the generator, only the Kingman's coalescent is represented by the term Φ_{ij} . To extend the process, we introduce more general coalescent.

Fix n and $f \in B(E^n)$. For $\mu \in M_1(E)$ define

$$\mathbf{B}G_{n,f}(\mu) = \sum_{\pi \in \mathcal{P}_n \setminus \{\mathbf{0}_{[n]}\}} \beta_\pi \left[\langle \mu^{|\pi|}, \Phi_\pi f \rangle - \langle \mu^n, f \rangle \right]. \quad (3.3)$$

For $f \in \mathcal{D}(A^{(n)})$ let L be the linear operator from \mathcal{D}_1 to $C(M_1(E))$ defined by

$$LG_{n,f}(\mu) = L_0 G_{n,f}(\mu) + \mathbf{B}G_{n,f}(\mu). \quad (3.4)$$

Define test functions $F(\mu) = \prod_{i=1}^n \langle \mu, f_i \rangle$ for any n and any $f_i \in B(E), i = 1, \dots, n$. The generator related to the simultaneous multiple part of the Ξ -coalescent is defined as

$$\mathbf{B}F(\mu) = \sum_{\pi \in \mathcal{P}_n \setminus \{\mathbf{0}_{[n]}\}} \beta_\pi \left(\prod_{i=1}^{|\pi|} \left\langle \mu, \prod_{j \in \pi_i} f_j \right\rangle - \prod_{i=1}^n \langle \mu, f_i \rangle \right).$$

The following Lemma shows a different representation for this generator, which has been obtained in Birkner et al [6] (Equation (4.2)) for jump type generator A . Its proof is essentially the same as [6] and we omit it.

Lemma 3.2 *The generator \mathbf{B} can be represented as*

$$\mathbf{B}F(\mu) = \int_{\Delta} \left(\sum_{i=1}^{\infty} z_i^2 \right)^{-1} \Xi_0(dz) \int_{E^{\mathbb{N}}} \left(F \left(\sum_{j=1}^{\infty} z_j \delta_{x_j} + \left(1 - \sum_{j=1}^{\infty} z_j \right) \mu \right) - F(\mu) \right) \mu^{\otimes \mathbb{N}}(dx),$$

where $x_i, i = 1, 2, 3, \dots$ are independently and identically distributed random variables with common distribution μ . Δ is the infinite simplex satisfying $\Delta = \{z = (z_1, z_2, \dots) : z_1 \geq z_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} z_i \leq 1\}$.

Lemma 3.2 shows that the (Ξ, A) -Fleming-Viot process is a jump diffusion type superprocess such that between jumping times it evolves like a classical Fleming-Viot process, and at a jumping time a fraction of its mass is redistributed over the current support of the process.

We now consider the martingale problem (MP) related to the SPDE to be considered later in Section 5. Let \mathcal{X} be the closure of $D(A) \subset C_b(E)$ with respect to the norm

$$\|f\| \equiv \|f\|_\infty + \|Af\|_\infty,$$

where $\|f\|_\infty$ is the supremum norm of f . Denote the dual space of the Banach space \mathcal{X} by \mathcal{X}^* . Introduce the distance in \mathcal{X}^* by

$$\rho(X, Y) = \sum_{j=1}^{\infty} 2^{-j} (|X(f_j) - Y(f_j)| \wedge 1),$$

where $\{f_j\} \subset D(A)$ is such that $\text{span}(\{f_j\})$ is dense in \mathcal{X} .

A stochastic process $X \in D(M_1(E))$ is a solution to the following generalized Fleming-Viot MP (GFVMP in short) if for any $f \in \mathcal{D}(A)$,

$$M_t^f = \langle X_t, f \rangle - \langle \mu, f \rangle - \int_0^t \langle X_s, Af \rangle ds \quad (3.5)$$

is a square-integrable martingale such that

$$\langle M^{f,c} \rangle_t = \sigma^2 \int_0^t (\langle X_s, f^2 \rangle - \langle X_s, f \rangle^2) ds, \quad (3.6)$$

and $\forall B \in \Gamma$, the process

$$\sum_{0 < s \leq t} 1_B(\Delta M_s) - \int_0^t \int_{\Delta} \int_{E^{\mathbb{N}}} 1_B \left(\sum_{i=1}^{\infty} z_i(\delta_{x_i} - X_s) \right) \gamma(dsdzdx) \quad (3.7)$$

is a martingale, where $M^{f,c}$ is the continuous part of the martingale M^f and M_t is the \mathcal{X}^* -valued martingale such that $M_t^f = M_t(f)$, $\Delta M_s = M_s - M_{s-}$,

$$\gamma(dsdzdx) = ds \otimes \left(\sum_{i=1}^{\infty} z_i^2 \right)^{-1} \Xi_0(dz) \otimes X_s^{\mathbb{N}}(dx)$$

is a random measure on $\mathbb{R}_+ \times \Delta \times E^{\mathbb{N}}$, and

$$\Gamma = \left\{ B \in \mathcal{B}(\mathcal{X}^* \setminus \{0\}) : \forall t > 0, \mathbb{E} \int_0^t \int_{\Delta} \int_{E^{\mathbb{N}}} 1_B \left(\sum_{i=1}^{\infty} z_i(\delta_{x_i} - X_s) \right) \gamma(dsdzdx) < \infty \right\}.$$

Theorem 3.3 *The GFVMP has a solution in $D_{M_1}[0, \infty)$. Further, every solution X to the GFVMP is a (Ξ, A) -Fleming-Viot process. Consequently, the GFVMP is well-posed.*

Proof. If X is a solution to the GFVMP, then

$$\sum_{0 < s \leq t} \langle \Delta M_s, f \rangle - \int_0^t \int_{\Delta} \int_{E^{\mathbb{N}}} \sum_{i=1}^{\infty} z_i(f(x_i) - \langle X_s, f \rangle) \gamma(dsdzdx)$$

is a pure jump martingale. Then (3.1) follows from Itô's formula (Theorem 4.57 of [15]), Lemma 3.2 and the martingale duality argument. The solution X is thus a (Ξ, A) -Fleming-Viot process.

Now we proceed to prove the existence of a solution for this martingale problem. First, we suppose that $(\sum_{i=1}^{\infty} z_i^2)^{-1} \Xi_0(dz)$ is a finite measure. Without loss of generality, we may assume that the total mass is 1. Let $0 < \tau_1 < \tau_2 < \dots$ be the jump times of a standard Poisson process. Let X_t be an $M_1(E)$ -valued process defined as follows. Between the Poisson times, it evolves like the usual Fleming-Viot process. At the jump time τ_j , we independently choose a Δ -valued random variable (z_i) with distribution $(\sum_{i=1}^{\infty} z_i^2)^{-1} \Xi_0$, and an $E^{\mathbb{N}}$ -valued random variable (x_i) with distribution $X_{\tau_j-}^{\otimes \mathbb{N}}$, $j = 1, 2, \dots$ and then set

$$X_{\tau_j} := \sum_{i=1}^{\infty} z_i \delta_{x_i} + \left(1 - \sum_{i=1}^{\infty} z_i\right) X_{\tau_j-}.$$

It is then easy to verify that X_t is a solution to the martingale problem (3.5).

In general, we approximate Ξ_0 by $\nu_n(dz) = 1_{\sum_{i=1}^{\infty} z_i^2 > 1/n} \Xi_0(dz)$. Let X^n be the solution to the GFVMP with Ξ_0 replaced by ν_n . Then,

$$M_t^n = X_t^n - \mu - \int_0^t A^* X_s^n ds$$

is a sequence of \mathcal{X}^* -valued martingales. Denote its continuous and purely-discontinuous parts by $M^{n,c}$ and $M^{n,d}$, respectively. Then

$$\langle M^{n,c}(f) \rangle_t = \sigma^2 \int_0^t \left(\langle X_s^n, f^2 \rangle - \langle X_s^n, f \rangle^2 \right) ds$$

and for any $B \in \Gamma$, the process

$$\sum_{0 < s \leq t} 1_B(\Delta M_s^n) - \int_0^t \int_{\Delta} \int_{E^{\mathbb{N}}} 1_B \left(\sum_{i=1}^{\infty} z_i (\delta_{x_i} - X_s^n) \right) \gamma^n(dsdzdx)$$

is a martingale, where A^* is the adjoint operator of A . Note that X^n is a sequence of \mathcal{X}^* -valued processes. To prove the tightness of X^n , we only need to prove the tightness of $\langle X^n, f \rangle$ for any $f \in \mathcal{X}$. Note that the finite variation part is

$$A_t^n = \int_0^t \langle X_s^n, Af \rangle ds$$

and the martingale part $M_t^n(f)$ has quadratic variation processes

$$\begin{aligned} \langle M^n(f) \rangle_t &= (\sigma^2 + \Xi_0(\Delta)) \int_0^t \left(\langle X_s^n, f^2 \rangle - \langle X_s^n, f \rangle^2 \right) ds \\ &\quad + \int_0^t \int_{\Delta} \int_{E^{\mathbb{N}}} \left(\sum_{i=1}^{\infty} z_i (f(x_i) - X_s^n(f)) \right)^2 \gamma^n(dsdzdx). \end{aligned}$$

It is easy to prove that $\{A^n\}$ and $\{\langle M^n(f) \rangle\}$ are C -tight. By Corollary 3.33 (p317) and Theorem 4.13 (p322) of Jacod and Shiryaev [15] (see also Theorem 6.1.1 of Kallianpur and Xiong [16]) we see that $\langle X^n, f \rangle$ is tight. So, X^n is tight. Similarly, we can prove the tightness of $\{(M^{n,c}, M^{n,d})\}$.

Denote a limit of $\{(X^n, M^{n,c}, M^{n,d})\}$ by $(X, \tilde{M}^1, \tilde{M}^2)$. It is easy to prove that $\tilde{M}^1(f)$ is a continuous martingale, $M_t = \tilde{M}_t^1 + \tilde{M}_t^2$ and

$$\left\langle \tilde{M}^1(f) \right\rangle_t = \sigma^2 \int_0^t \left(\langle X_s, f^2 \rangle - \langle X_s, f \rangle^2 \right) ds.$$

By the same arguments as those in the proofs of Theorem 6.1.3 and Lemma 6.1.11 of [16], we can prove that $\{\tilde{M}^2(f)\}$ is purely-discontinuous, and for any $B \in \Gamma$, the process

$$\sum_{0 < s \leq t} 1_B(\Delta \tilde{M}_s^2) - \int_0^t \int_{\Delta} \int_{E^{\mathbb{N}}} 1_B \left(\sum_{i=1}^{\infty} z_i \delta_{x_i} - \sum_{i=1}^{\infty} z_i X_s \right) \gamma(dsdzdx)$$

is a martingale. Thus, (X_t) is a solution to the martingale problem (3.5). \square

4 The reversibility

In this section, we consider the reversibility of the (Ξ, A) -Fleming-Viot process whose existence is justified in the previous section. We will prove that it is irreversible except for the case of classical Fleming-Viot process with parent independent mutation.

To obtain the existence and uniqueness of the invariant measure for the (Ξ, A) -Fleming-Viot process we need the following assumption.

Assumption (I): The Markov process Z with generator A has a unique invariant measure $\nu \in M_1(E)$ such that $P_t^* \mu \rightarrow \nu$ weakly for any $\mu \in M_1(E)$ as $t \rightarrow \infty$, where (P_t^*) denotes the adjoint for (P_t) .

Lemma 4.1 *Suppose that the Assumption (I) holds. Then,*

- (a) *The (Ξ, A) -Fleming-Viot process X has at least one invariant measure.*
- (b) *Let Π be any invariant measure of the (Ξ, A) -Fleming-Viot process X . For any positive integer n and any $f \in B(E^n)$, we have*

$$\int_{M_1(E)} \langle \mu^n, f \rangle \Pi(d\mu) = \mathbb{E}_{(n,f)} \langle \nu, Z_\tau \rangle,$$

where

$$\tau := \inf\{t \geq 0 : |M_t| = 1\}$$

and (M, Z) is the (Ξ, A) -coalescent starting at (n, f) . Consequently, X has a unique invariant measure.

Proof. (a) As $P_t^* X_0 \rightarrow \nu$, the family $\{P_t^* X_0 : t \geq 0\}$ is pre-compact, and hence, tight in $M_1(E)$. Thus, for any $\epsilon > 0$, there exists a compact subset K_ϵ of E such that $P_t^* X_0(K_\epsilon^c) < \epsilon$ for all $t \geq 0$. Let

$$\mathbf{K}_\epsilon = \{\rho \in M_1(E) : \rho(K_{\epsilon k^{-1} 2^{-k}}^c) \leq k^{-1}, \forall k \geq 1\}.$$

For any $\delta > 0$, let $k \geq 1$ be such that $k^{-1} < \delta$. Then, for any $\rho \in \mathbf{K}_\epsilon$, we have

$$\rho(K_{\epsilon k^{-1} 2^{-k}}^c) \leq k^{-1} < \delta,$$

and hence, \mathbf{K}_ϵ is tight in $M_1(E)$. Then, \mathbf{K}_ϵ is a pre-compact subset of $M_1(E)$.

Note that

$$\begin{aligned}
T^{-1} \int_0^T P X_t^{-1} dt (\mathbf{K}_\epsilon^c) &= T^{-1} \int_0^T P (\exists k \geq 1, X_t(K_{\epsilon k^{-1} 2^{-k}}^c) > k^{-1}) dt \\
&\leq T^{-1} \int_0^T \sum_{k=1}^{\infty} k \mathbb{E} X_t(K_{\epsilon k^{-1} 2^{-k}}^c) dt \\
&= T^{-1} \int_0^T \sum_{k=1}^{\infty} k P_t^* X_0(K_{\epsilon k^{-1} 2^{-k}}^c) dt \\
&< T^{-1} \int_0^T \sum_{k=1}^{\infty} k \epsilon k^{-1} 2^{-k} dt = \epsilon.
\end{aligned}$$

Thus, the family $\left\{ T^{-1} \int_0^T P X_t^{-1} dt : T \geq 0 \right\}$ is tight, and hence, pre-compact in $M_1(M_1(E))$. Let Π be a limit point. Then there exists a sequence (T_n) such that $T_n \uparrow \infty$ and

$$\lim_{n \rightarrow \infty} T_n^{-1} \int_0^{T_n} P X_t^{-1} dt = \Pi.$$

For any $s \geq 0$,

$$\begin{aligned}
\Pi X_s^{-1} &= \lim_{n \rightarrow \infty} T_n^{-1} \int_0^{T_n} P X_t^{-1} \circ X_s^{-1} dt \\
&= \lim_{n \rightarrow \infty} T_n^{-1} \int_s^{T_n+s} P X_t^{-1} dt \\
&= \Pi.
\end{aligned}$$

Namely, Π is an invariant measure of the stochastic process $\{X_t\}$.

(b) Note that $\tau < \infty$. By the moment duality (3.1) and the strong Markov property we have

$$\begin{aligned}
\int_{M_1(E)} \langle \mu^n, f \rangle \Pi(d\mu) &= \lim_{t \rightarrow \infty} \mathbb{E}_\Pi \langle X_t^n, f \rangle \\
&= \lim_{t \rightarrow \infty} \int_{M_1(E)} \mathbb{E}_{(n,f)} [G_\mu(M_t, Z_t)] \Pi(d\mu) \\
&= \lim_{t \rightarrow \infty} \int_{M_1(E)} \mathbb{E}_{(n,f)} [G_\mu(M_t, Z_t) 1_{\tau \leq t}] \Pi(d\mu) \\
&= \lim_{t \rightarrow \infty} \int_{M_1(E)} \mathbb{E}_{(n,f)} \mathbb{E}_{(1,Z_\tau)} [\langle \mu, Z_{t-\tau} \rangle 1_{\tau \leq t}] \Pi(d\mu) \\
&= \lim_{t \rightarrow \infty} \int_{M_1(E)} \mathbb{E}_{(n,f)} \mathbb{E}_{(1,Z_\tau)} \langle \mu, Z_t \rangle \Pi(d\mu) \\
&= \lim_{t \rightarrow \infty} \int_{M_1(E)} \mathbb{E}_{(n,f)} \langle \mu, P_t Z_\tau \rangle \Pi(d\mu) \\
&= \lim_{t \rightarrow \infty} \int_{M_1(E)} \mathbb{E}_{(n,f)} \langle P_t^* \mu, Z_\tau \rangle \Pi(d\mu) \\
&= \mathbb{E}_{(n,f)} \langle \nu, Z_\tau \rangle,
\end{aligned}$$

where we have used the fact that $\tau < \infty$ a.s.. □

We now consider the reversibility of the (Ξ, A) -Fleming-Viot process X .

The transition semigroup (P_t) is irreducible if for any $x \in E$ and $f \in C_0^+(E)$ with $f \neq 0$, $P_t f(x) > 0$ for some $t > 0$, where $C_0^+(E)$ denotes the space of nonnegative and continuous functions vanishing at infinity.

Lemma 4.2 *Suppose that (P_t) is irreducible. If the (Ξ, A) -Fleming-Viot process X is reversible, then its mutation generator A is a parent independent pure jump generator, i.e.*

$$Af(\cdot) = \frac{\theta}{2} \int_E (f(y) - f(\cdot)) \nu_0(dy)$$

for some $\theta > 0$ and probability measure ν_0 on E .

Proof. The result of the current Lemma has been proved for the classical Fleming-Viot process by Li et al [18], Handa [12] and Schmuland and Sun [24] with different methods. It is simple to see that the barycenter m of a stationary distribution of the classical Fleming-Viot process is a stationary distribution for its mutation operator A . A key observation in [18] is that the reversibility of A relative to m is structured by the first three moment measures of the classical Fleming-Viot process. The necessary form of A was then derived in [18] from Beurling-Deny formula of the associated Dirichlet form.

We shall see that the structures explained above are maintained by the generalized Fleming-Viot processes. Our proof is an adaption of the approach for Theorem 1.1 of [18]. The results of Lemmas 2.1, 2.4, 2.5 and 2.6 in [18] still hold for the generalized Fleming-Viot process. We only need to make some modifications in the first half of the proof of their Lemma 2.1 and the second half of the proof of their Lemma 2.5 as follows. By possible time rescaling, without loss of generality we assume $\sigma^2 + \beta_{2;2;0} = 1$ in the sequel of this proof.

For the proof of the analog of Lemma 2.1 of [18], let Π be the reversible measure of X and define the moment measures

$$m := \mathbb{E}_\Pi X_t, \quad m_2 := \mathbb{E}_\Pi X_t \otimes X_t \quad \text{and} \quad m_3 := \mathbb{E}_\Pi X_t \otimes X_t \otimes X_t,$$

where we adopt the notation of [18]. Given any partition $0 = t_0 < t_1 < \dots < t_n = t$ and $f, g, h \in \mathcal{D}(A)$, by the invariance of Π and the moment duality we have

$$\begin{aligned} & m_2(P_{t_i} f \otimes P_{t_{i+1}+r} g) \\ &= \mathbb{E}_\Pi [X_{t_{i+1}-t_i}(P_{t_i} f) X_{t_{i+1}-t_i}(P_{t_{i+1}+r} g)] \\ &= e^{-(\sigma^2 + \beta_{2;2;0})(t_{i+1}-t_i)} \mathbb{E}_\Pi [X_0(P_{t_{i+1}} f) X_0(P_{t_{i+1}+r} g)] \\ &\quad + \int_0^{t_{i+1}-t_i} ds (\sigma^2 + \beta_{2;2;0}) e^{-(\sigma^2 + \beta_{2;2;0})s} \mathbb{E}_\Pi X_0(P_{t_{i+1}-t_i-s}(P_{t_i+s} f P_{t_i+s+r} g)) \\ &= e^{-(t_{i+1}-t_i)} m_2(P_{t_{i+1}} f \otimes P_{t_{i+1}+r} g) + \int_0^{t_{i+1}-t_i} ds e^{-s} m(P_{t_{i+1}-t_i-s}(P_{t_i+s} f P_{t_i+s+r} g)). \end{aligned}$$

Letting $n \rightarrow \infty$ and $t_{i+1} - t_i = t/n$, we have

$$\begin{aligned} & m_2(f \otimes P_r g) - m_2(P_t f \otimes P_{t+r} g) \\ &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (m_2(P_{t_i} f \otimes P_{t_i+r} g) - m_2(P_{t_{i+1}} f \otimes P_{t_{i+1}+r} g)) \\ &= \int_0^t m(P_s f P_{s+r} g) ds - \int_0^t m_2(P_s f \otimes P_{s+r} g) ds. \end{aligned}$$

We have thus obtained (2.4) of [18].

For the proof of the analog of Lemma 2.5 of [18] by the reversibility and the moment duality again,

$$\begin{aligned}
& \mathbb{E}_\Pi[X_0^{\otimes 2}(f \otimes g)X_0(P_th - h)] \\
&= \int \Pi(d\mu)\mathbb{E}_\mu[(X_t^{\otimes 2}(f \otimes g) - X_0^{\otimes 2}(f \otimes g))X_0(h)] \\
&= \int \Pi(d\mu)\mathbb{E}_\mu \left[X_0(h) \left(e^{-(\sigma^2 + \beta_{2;2;0})t} X_0^{\otimes 2}(P_tf \otimes P_tg) \right. \right. \\
&\quad \left. \left. + \int_0^t ds(\sigma^2 + \beta_{2;2;0})e^{-(\sigma^2 + \beta_{2;2;0})s} X_0(P_{t-s}(P_sfP_sg))ds - X_0^{\otimes 2}(f \otimes g) \right) \right].
\end{aligned} \tag{4.1}$$

Dividing both sides of (4.1) by t and letting $t \rightarrow 0+$, we have

$$\begin{aligned}
& \mathbb{E}_\Pi[X_0^{\otimes 3}(f \otimes g \otimes Ah)] \\
&= \mathbb{E}_\Pi[X_0^{\otimes 3}(Af \otimes g \otimes h + f \otimes Ag \otimes h - (\sigma^2 + \beta_{2;2;0})f \otimes g \otimes h) + X_0^{\otimes 2}((\sigma^2 + \beta_{2;2;0})(fg) \otimes h)] \\
&= \mathbb{E}_\Pi[X_0^{\otimes 3}(Af \otimes g \otimes h + f \otimes Ag \otimes h - f \otimes g \otimes h) + X_0^{\otimes 2}((fg) \otimes h)].
\end{aligned}$$

So,

$$m_3(f \otimes g \otimes Ah) = m_3(Af \otimes g \otimes h + f \otimes Ag \otimes h - f \otimes g \otimes h) - m_2((fg) \otimes h).$$

The rest of the proof then follows from that in Lemma 2.5 of [18].

The same argument for Theorem 1.1 of [18] can also go through for the generalized Fleming-Viot process. \square

Lemma 4.3 *If $\beta_{p;p;0} = 0$ for some $p \geq 3$, then the Ξ -coalescent degenerates to Kingman's coalescent.*

Proof. For any $p \geq 3$, by (2.1) we have

$$\beta_{p;p;0} = \int_{\Delta} \left(\sum_{i=1}^{\infty} x_i^p \right) \left(\sum_{i=1}^{\infty} x_i^2 \right)^{-1} \Xi_0(d\mathbf{x}).$$

If $\beta_{p;p;0} = 0$, then Ξ_0 must be a zero measure since Ξ_0 has no atom at zero, and the integrand is always positive except at zero. Therefore, the Ξ -coalescent degenerates to Kingman's coalescent. \square

Theorem 4.4 *Suppose that (P_t) is irreducible and space E contains at least three different points or $E = \{e_1, e_2\}$ with $\nu_0(\{e_1\}) \neq \nu_0(\{e_2\})$. Then the (Ξ, A) -Fleming-Viot process X is reversible if and only if it is the classical Fleming-Viot process with parent independent mutation.*

Proof. If the generalized Fleming-Viot process X is reversible, then there exists an invariant measure, say Π , satisfying

$$\int_{M_1(E)} G_{p,f}(\mu) LG_{q,g}(\mu) \Pi(d\mu) = \int_{M_1(E)} G_{q,g}(\mu) LG_{p,f}(\mu) \Pi(d\mu), \tag{4.2}$$

where p and q are arbitrary nonnegative integers.

By Lemma 4.2 the generator A is a parent independent jump operator. Also note that ν_0 has a support E since (P_t) is irreducible. If E contains at least three different points or $E = \{e_1, e_2\}$ with $\nu_0(\{e_1\}) \neq \nu_0(\{e_2\})$, we can choose $F \subset E$ such that

$$\nu_0(F) = \alpha \neq 0, 1/2, 1.$$

Taking $p = 1$, $q = 0$ and $f = 1_F$ in (4.2) we get

$$\int_{M_1(E)} \mu(F) \Pi(d\mu) = \alpha. \quad (4.3)$$

Taking $p = 2$, $q = 0$ and $f = 1_{F \times F}$, we get

$$\int_{M_1(E)} \mu^2(F) \Pi(d\mu) = \frac{\sigma^2 + \beta_{2;2;0} + \theta\alpha}{\sigma^2 + \beta_{2;2;0} + \theta} \alpha. \quad (4.4)$$

Taking $p = 3$, $q = 0$ and $f = 1_{F \times F \times F}$, we get

$$\int_{M_1(E)} \mu^3(F) \Pi(d\mu) = \frac{\beta_{3;3;0}\alpha + \frac{3}{2}(2\sigma^2 + 2\beta_{3;2;1} + \theta\alpha) \frac{\sigma^2 + \beta_{2;2;0} + \theta\alpha}{\sigma^2 + \beta_{2;2;0} + \theta}}{3\sigma^2 + 3\beta_{3;2;1} + \beta_{3;3;0} + \frac{3}{2}\theta} \alpha. \quad (4.5)$$

Taking $p = 1$, $q = 2$, $f = 1_F$ and $g = 1_{F \times F}$, we get

$$\int_{M_1(E)} \mu^2(F) \Pi(d\mu) \times \left(\beta_{2;2;0} + \sigma^2 + \frac{\theta\alpha}{2} \right) - \int_{M_1(E)} \mu^3(F) \Pi(d\mu) \times \left(\beta_{2;2;0} + \sigma^2 + \frac{\theta}{2} \right) = 0. \quad (4.6)$$

Plugging in Equations (4.4) and (4.5). Since $\beta_{3;2;1} = \beta_{2;2;0} - \beta_{3;3;0}$ (consistency condition), then Equation (4.6) is equivalent to

$$\frac{\beta_{3;3;0}\theta^2\alpha(2\alpha - 1)(\alpha - 1)}{(\sigma^2 + \beta_{2;2;0} + \theta)(6\sigma^2 + 6\beta_{2;2;0} - 4\beta_{3;3;0} + 3\theta)} = 0. \quad (4.7)$$

Note that

$$6\sigma^2 + 6\beta_{2;2;0} - 4\beta_{3;3;0} + 3\theta = 6\sigma^2 + 6\beta_{3;2;1} + 2\beta_{3;3;0} + 3\theta > 0,$$

$\theta > 0$ and $\alpha \neq 0, 1/2, 1$. Consequently, we have $\beta_{3;3;0} = 0$. By Lemma 4.3 we have $\Xi_0 = 0$. Hence, X_t becomes the usual Fleming-Viot process with parent independent mutation, which is known to be reversible. \square

In the following we consider the reversibility for several classes of (Ξ, A) -Fleming-Viot processes under the condition $E = \{e_1, e_2\}$ with $\nu_0(\{e_1\}) = 1/2 = \nu_0(\{e_2\})$. The proofs will be given in the Appendix.

For $\beta \in (0, 2)$, the $(\text{Beta}(2-\beta, \beta), A)$ -Fleming-Viot process is the (Ξ, A) -Fleming-Viot process with measure Ξ supported on $[0, 1]$ satisfying

$$\Xi(dx) = \frac{\Gamma(2)}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta} (1-x)^{\beta-1} dx. \quad (4.8)$$

Proposition 4.5 *Given any parent independent pure jump generator A on $E = \{e_1, e_2\}$ with $\nu_0(\{e_1\}) = \nu_0(\{e_2\}) = \alpha = 1/2$, the $(\text{Beta}(2-\beta, \beta), A)$ -Fleming-Viot process is not reversible.*

Proposition 4.6 *Given any parent independent pure jump generator A on $E = \{e_1, e_2\}$ with $\nu_0(\{e_1\}) = \nu_0(\{e_2\}) = \alpha = 1/2$, the (Ξ, A) -Fleming-Viot process with the finite measure Ξ on $[0, 1]$ defined as*

$$\Xi(dx) = x^{-\gamma} dx$$

for some $\gamma \in (0, 1)$ is not reversible.

Proposition 4.7 *Given any parent independent pure jump generator A on $E = \{e_1, e_2\}$ with $\nu_0(\{e_1\}) = \nu_0(\{e_2\}) = \alpha = 1/2$, the (Ξ, A) -Fleming-Viot process with $\Xi = \delta_1$, the point mass on $[0, 1]$, is not reversible.*

Sagitov [22] introduced the Poisson-Dirichlet coalescent with parameter ϵ as follows. Let Ξ be the Poisson-Dirichlet distribution $\Pi_\epsilon(dx)$ with a positive parameter ϵ on the infinite simplex

$$\Delta^* = \left\{ \mathbf{x} \in \Delta : \sum_{i=1}^{\infty} x_i = 1 \right\}.$$

Then the Π_ϵ -coalescence rates are

$$\lambda_{b; k_1, \dots, k_r; s} = \frac{\epsilon^{r+s}}{\epsilon^{[b]}} \prod_{i=1}^r (k_i - 1)! \quad (4.9)$$

for all $r \geq 1$, $k_1, \dots, k_r \geq 2$ and $s \geq 0$, where $b = k_1 + \dots + k_r + s$ and

$$\epsilon^{[b]} = \epsilon(\epsilon + 1) \cdots (\epsilon + b - 1)$$

is the ascending factorial power.

Proposition 4.8 *Given a parent independent pure jump generator A on $E = \{e_1, e_2\}$ with $\nu_0(\{e_1\}) = \nu_0(\{e_2\}) = \alpha = 1/2$, the (Ξ, A) -Fleming-Viot process with $\Xi = \Pi_\epsilon$ is not reversible.*

Remark 4.9 *For $E = \{e_1, e_2\}$ with $\nu_0(\{e_1\}) = 1/2 = \nu_0(\{e_2\})$, we conjecture that the Ξ -Fleming-Viot process is reversible only if it degenerates to the classical Fleming-Viot process.*

5 An SPDE for the $(\Xi, \frac{1}{2}\Delta)$ -Fleming-Viot process

In this section we switch to a different topic and consider the weak and strong uniqueness for solution to an SPDE associated to the $(\Xi, \frac{1}{2}\Delta)$ -Fleming-Viot process, where $\Delta = \frac{d^2}{dx^2}$ denotes the Laplacian mutation generator. Note that Δ is also used to represent the set of infinite dimensional simplex in this paper. Throughout this section we assume the type space E to be the real line \mathbb{R} .

Next, we hope to characterize the $(\Xi, \frac{1}{2}\Delta)$ -Fleming-Viot process by an SPDE which has strong uniqueness. Here we adapt the approach of Xiong [25]. For a measure X , we define the distribution function

$$u(x) = X((-\infty, x]), \quad \forall x \in \mathbb{R}. \quad (5.1)$$

Let $u_{s-}^{-1}(y) = \inf\{x : u_{s-}(x) \geq y\}$. Consider the following SPDE

$$u_t(x) = u_0(x) + \sigma \int_0^t \int_0^1 (1_{y \leq u_{s-}(x)} - u_{s-}(x)) B(ds dy) + \int_0^t \frac{1}{2} \Delta u_{s-}(x) ds$$

$$+ \int_0^t \int_{\Delta} \int_{[0,1]^{\mathbb{N}}} \left(\sum_{i=1}^{\infty} z_i 1_{y_i \leq u_{s-}(x)} - \sum_{i=1}^{\infty} z_i u_{s-}(x) \right) M(ds dz dy), \quad (5.2)$$

where $B(ds dy)$ is a white noise on $\mathbb{R}_+ \times (0, 1]$ with intensity $ds dy$ and $M(ds dz dy)$ is an independent Poisson random measure on $\mathbb{R}_+ \times \Delta \times [0, 1]^{\mathbb{N}}$ with intensity $ds \left(\sum_{i=1}^{\infty} z_i^2 \right)^{-1} \Xi_0(dz)(dy)^{\otimes \mathbb{N}}$.

Denote $\langle f, g \rangle := \int_{\mathbb{R}} f(x)g(x)dx$ for functions f and g on \mathbb{R} . We say an \mathbb{R} -valued random field $u = \{u_t(x) : t \geq 0, x \in \mathbb{R}\}$ is a solution to the SPDE (5.2) if for any $f \in C_0^2(\mathbb{R})$, the collection of compactly supported functions on \mathbb{R} with continuous second derivatives,

$$\begin{aligned} \langle u_t, f \rangle &= \langle u_0, f \rangle + \sigma \int_0^t \int_0^1 \int_{\mathbb{R}} (1_{y \leq u_{s-}(x)} - u_{s-}(x)) f(x) dx B(ds dy) + \int_0^t \left\langle u_{s-}, \frac{1}{2} f'' \right\rangle ds \\ &\quad + \int_0^t \int_{\Delta} \int_{[0,1]^{\mathbb{N}}} \int_{\mathbb{R}} \left(\sum_{i=1}^{\infty} z_i 1_{y_i \leq u_{s-}(x)} - \sum_{i=1}^{\infty} z_i u_{s-}(x) \right) f(x) dx M(ds dz dy). \end{aligned}$$

A weak solution to the SPDE (5.2) is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration (\mathcal{F}_t) and adapted processes (u, B, M) such that (5.2) holds.

Theorem 5.1 *The SPDE (5.2) has a weak solution. If the random field $u_t(x)$ is a solution to the SPDE (5.2), then the measure-valued process X defined by (5.1) is a $(\Xi, \frac{1}{2}\Delta)$ -Fleming-Viot process. Consequently, the SPDE (5.2) has a unique weak solution.*

Proof. Suppose that the SPDE (5.2) has a solution. For any $f \in C_0^3(\mathbb{R})$, we have

$$\begin{aligned} \langle X_t, f \rangle &= -\langle u_t, f' \rangle \\ &= -\langle u_0, f' \rangle - \sigma \int_0^t \int_0^1 \int_{\mathbb{R}} (1_{y \leq u_{s-}(x)} - u_{s-}(x)) f'(x) dx B(ds dy) - \int_0^t \left\langle u_{s-}, \frac{1}{2} f''' \right\rangle ds \\ &\quad - \int_0^t \int_{\Delta} \int_{[0,1]^{\mathbb{N}}} \int_{\mathbb{R}} \left(\sum_{i=1}^{\infty} z_i 1_{y_i \leq u_{s-}(x)} - \sum_{i=1}^{\infty} z_i u_{s-}(x) \right) f'(x) dx M(ds dz dy) \\ &= \langle X_0, f \rangle - \sigma \int_0^t \int_0^1 \left(\int_{u_{s-}^{-1}(y)}^{\infty} df(x) - \langle u_{s-}, f' \rangle \right) B(ds dy) + \int_0^t \left\langle X_{s-}, \frac{1}{2} f'' \right\rangle ds \\ &\quad - \int_0^t \int_{\Delta} \int_{[0,1]^{\mathbb{N}}} \left(\sum_{i=1}^{\infty} z_i \int_{u_{s-}^{-1}(y_i)}^{\infty} df(x) - \sum_{i=1}^{\infty} z_i \langle u_{s-}, f' \rangle \right) M(ds dz dy) \\ &= \langle X_0, f \rangle + \sigma \int_0^t \int_0^1 (f(u_{s-}^{-1}(y)) - \langle X_{s-}, f \rangle) B(ds dy) + \int_0^t \left\langle X_{s-}, \frac{1}{2} f'' \right\rangle ds \\ &\quad + \int_0^t \int_{\Delta} \int_{[0,1]^{\mathbb{N}}} \left(\sum_{i=1}^{\infty} z_i f(u_{s-}^{-1}(y_i)) - \sum_{i=1}^{\infty} z_i \langle X_{s-}, f \rangle \right) M(ds dz dy). \end{aligned}$$

Thus,

$$M_t(f) = \langle X_t, f \rangle - \langle X_0, f \rangle - \int_0^t \left\langle X_{s-}, \frac{1}{2} f'' \right\rangle ds$$

is a square-integrable martingale with continuous part

$$M_t^c(f) = \sigma \int_0^t \int_0^1 (f(u_{s-}^{-1}(y)) - \langle X_{s-}, f \rangle) B(ds dy)$$

and pure jump part

$$M_t^d(f) = \int_0^t \int_{\Delta} \int_{[0,1]^{\mathbb{N}}} \left(\sum_{i=1}^{\infty} z_i f(u_{s-}^{-1}(y_i)) - \sum_{i=1}^{\infty} z_i \langle X_{s-}, f \rangle \right) M(ds dz dy).$$

Then

$$\begin{aligned} \langle M^c(f) \rangle_t &= \sigma^2 \int_0^t \int_0^1 (f(u_{s-}^{-1}(y)) - \langle X_{s-}, f \rangle)^2 dy ds \\ &= \sigma^2 \int_0^t \int_0^1 (f(u_{s-}^{-1}(y))^2 - 2 \langle X_{s-}, f \rangle f(u_{s-}^{-1}(y)) + \langle X_{s-}, f \rangle^2) dy ds \\ &= \sigma^2 \int_0^t (\langle X_{s-}, f^2 \rangle - \langle X_{s-}, f \rangle^2) ds. \end{aligned}$$

Further,

$$\sum_{0 < s \leq t} 1_B(\Delta M_s) - \int_0^t \int_{\Delta} \int_{[0,1]^{\mathbb{N}}} 1_B \left(\sum_{i=1}^{\infty} z_i (\delta_{u_{s-}^{-1}(y_i)} - X_{s-}) \right) \left(\sum_{i=1}^{\infty} z_i^2 \right)^{-1} \Xi_0(dz) dy ds$$

is a martingale. As

$$\begin{aligned} & \int_0^t \int_{\Delta} \int_{[0,1]^{\mathbb{N}}} 1_B \left(\sum_{i=1}^{\infty} z_i (\delta_{u_{s-}^{-1}(y_i)} - X_{s-}) \right) \left(\sum_{i=1}^{\infty} z_i^2 \right)^{-1} \Xi_0(dz) dy ds \\ &= \int_0^t \int_{\Delta} \int_{\mathbb{R}^{\mathbb{N}}} 1_B \left(\sum_{i=1}^{\infty} z_i (\delta_{x_i} - X_s) \right) \gamma(ds dz dx), \end{aligned}$$

we see that

$$\sum_{0 < s \leq t} 1_B(\Delta M_s) - \int_0^t \int_{\Delta} \int_{\mathbb{R}^{\mathbb{N}}} 1_B \left(\sum_{i=1}^{\infty} z_i (\delta_{x_i} - X_s) \right) \gamma(ds dz dx)$$

is a martingale. Therefore, X_t is the generalized Fleming-Viot process.

The existence of weak solution follows from Theorem 3.3. \square

Next, we consider the backward version with T fixed:

$$v_t(x) = u_{T-t}(x), \quad \forall t \in [0, T], \quad x \in \mathbb{R}.$$

Note that v_t is left-continuous with right limit. We also define the backward random measure

$$\tilde{B}([0, s] \times A) = B([T-s, T] \times A), \quad \forall s \in [0, T], \quad A \in \mathcal{B}([0, 1]).$$

The random measure \tilde{M} is defined similarly. Then, $\{v_t(x)\}$ satisfies the following backward SPDE:

$$\begin{aligned} v_t(x) &= u_0(x) + \sigma \int_t^T \int_0^1 (1_{y \leq v_{s+}(x)} - v_{s+}(x)) \tilde{B}(\hat{d}s dy) + \int_t^T \frac{1}{2} \Delta v_{s+}(x) ds \\ &\quad + \int_t^T \int_{\Delta} \int_{[0,1]^{\mathbb{N}}} \left(\sum_{i=1}^{\infty} z_i 1_{y_i \leq v_{s+}(x)} - \sum_{i=1}^{\infty} z_i v_{s+}(x) \right) \tilde{M}(\hat{d}s dz dy), \end{aligned} \quad (5.3)$$

where $\hat{d}s$ denotes the backward Itô's integral.

It is clear that if (5.3) has a unique solution, so does (5.2), and vice versa. To prove the uniqueness of the solution to (5.3), we adapt the idea of Xiong [25] by relating it to a backward triply stochastic differential equation (BTSDE). Denote

$$W_s^{t,x} = x + W_s - W_t, \quad \forall t \leq s \leq T,$$

where W_t is a one-dimensional standard Brownian motion.

Fix t and x , we consider the following BTSDE: For $t \leq s \leq T$,

$$\begin{aligned} Y_s^{t,x} &= u_0(W_T^{t,x}) + \sigma \int_s^T \int_0^1 \left(1_{y \leq Y_{r+}^{t,x}} - Y_{r+}^{t,x} \right) \tilde{B}(\hat{d}r dy) \\ &\quad + \int_s^T \int_{\Delta} \int_{[0,1]^{\mathbb{N}}} \left(\sum_{i=1}^{\infty} z_i 1_{y_i \leq Y_{r+}^{t,x}} - \sum_{i=1}^{\infty} z_i Y_{r+}^{t,x} \right) \tilde{M}(\hat{d}r dz dy) - \int_s^T Z_r^{t,x} dW_r. \end{aligned} \quad (5.4)$$

Let $\mathcal{G}_t = \sigma(\mathcal{G}_t^1, \mathcal{G}_t^2)$ where \mathcal{G}_t^1 (non-decreasing) and \mathcal{G}_t^2 (non-increasing) are independent sigma field families such that for any t , $\mathcal{F}_t^W \subset \mathcal{G}_t^1$ and $\mathcal{F}_{t,T}^{B,M} \subset \mathcal{G}_t^2$, where

$$\mathcal{F}_{t,T}^{B,M} = \sigma \left(\begin{array}{l} B([r, T] \times A_1), \quad M([r, T] \times A_2 \times A_3), \\ r \in [t, T], \quad A_1 \in \mathcal{B}([0, 1]), \quad A_2 \in \mathcal{B}(\Delta) \text{ and } A_3 \in \mathcal{B}([0, 1]^{\mathbb{N}}) \end{array} \right). \quad (5.5)$$

The solution $(Y_s^{t,x}, Z_s^{t,x})$ to the BTSDE (5.4) is \mathcal{G}_s -adapted.

To establish the connection between the backward SPDE (5.3) and the BTSDE (5.4), we need the following extended Itô's formula.

Lemma 5.2 *Let Y_t be a process given by*

$$Y_t = Y_T + \int_t^T \int_{U_1} G_1(r, u) B(\hat{d}r du) + \int_t^T \int_{U_2} G_2(r, v) \tilde{M}(\hat{d}r dv) - \int_t^T Z_r dW_r,$$

where B is a Gaussian white noise with intensity $\mu_1(du)ds$ and M is a Poisson random measure with intensity $\mu_2(dv)ds$. Further, $G_i : [0, T] \times U_i \times \Omega \rightarrow \mathbb{R}$, $i = 1, 2$, are \mathcal{G}_t adapted, where \mathcal{G}_t is defined by (5.5) with obvious modification. Then, for any $f \in C^2(\mathbb{R})$, we have

$$\begin{aligned} f(Y_t) &= f(Y_T) + \int_t^T \int_{U_1} f'(Y_s) G_1(s, u) B(\hat{d}r du) - \int_t^T f'(Y_s) Z_s dW_s \\ &\quad + \int_t^T \int_{U_2} (f(Y_{s+} + G_2(s, v)) - f(Y_{s+})) \tilde{M}(\hat{d}r dv) \\ &\quad + \int_t^T \int_{U_2} (f(Y_s + G_2(s, v)) - f(Y_s) - f'(Y_s) G_2(s, v)) \mu_2(dv) ds \\ &\quad + \frac{1}{2} \int_t^T \int_{U_1} f''(Y_s) G_1(s, u)^2 \mu_1(du) ds - \frac{1}{2} \int_t^T Z_s^2 f''(Y_s) ds. \end{aligned} \quad (5.6)$$

The proof of (5.6) follows from that of the standard Itô's formula for semimartingale driven by Brownian motion and Poisson random measure; see Chapter II of [14]. Here, we omit the detail.

Since the second order moments of X_t are the same as those for the classical Fleming-Viot process, by the same arguments as in the proof of Lemma 2.3 of Konno and Shiga [17] we can show that almost surely, X_t has a density for almost all t .

Let T_δ be the Brownian semigroup with p_δ the corresponding heat kernel.

Lemma 5.3 Suppose that X_0 has a density $\partial_x u_0 \in L^2(\mathbb{R}, e^{-|x|} dx)$. Then, there exists $h \in L^2([0, T] \times \mathbb{R}, e^{-2|x|} dx dt)$ such that

$$\lim_{\delta \rightarrow 0+} \mathbb{E} \int_0^T \int_{\mathbb{R}} (T_\delta X_t(x) - h_t(x))^2 e^{-2|x|} dx dt = 0.$$

Proof. For any $\delta, \delta' > 0$, by the moment duality we have for $\lambda = \Xi_0(\Delta)$,

$$\begin{aligned} & \mathbb{E} \int_0^T dt \int_{\mathbb{R}} (T_\delta X_t(x) - T_{\delta'} X_t(x))^2 e^{-2|x|} dx \\ &= \int_0^T dt \int_{\mathbb{R}} \mathbb{E} \left(\int_{\mathbb{R}} p_\delta(x-y) X_t(dy) - \int_{\mathbb{R}} p_{\delta'}(x-y) X_t(dy) \right)^2 e^{-2|x|} dx \\ &= \int_0^T dt \int_{\mathbb{R}} e^{-2|x|} dx \int_0^t \lambda e^{-\lambda s} ds \int_{\mathbb{R}} X_0(dy) \int_{\mathbb{R}} p_{t-s}(y-z) (p_{s+\delta}(x-z) - p_{s+\delta'}(x-z))^2 dz \\ & \quad + \int_0^T e^{-\lambda t} dt \int_{\mathbb{R}} e^{-2|x|} dx \left(\int_{\mathbb{R}} X_0(dy) p_{t+\delta}(x-y) - \int_{\mathbb{R}} X_0(dy) p_{t+\delta'}(x-y) \right)^2. \end{aligned}$$

For $X_0(dy) = X_0(y) dy$,

$$\begin{aligned} & \int_{\mathbb{R}} e^{-2|x|} dx \left(\int_{\mathbb{R}} (p_{t+\delta}(x-y) - p_{t+\delta'}(x-y)) X_0(y) dy \right)^2 \\ & \leq \int_{\mathbb{R}} e^{-2|x|} dx \int_{\mathbb{R}} (p_{t+\delta}(x-y) - p_{t+\delta'}(x-y))^2 e^{|y|} dy \int_{\mathbb{R}} X_0(y)^2 e^{-|y|} dy. \end{aligned}$$

We can show that

$$|p_{t+\delta}(z) - p_{t+\delta'}(z)| \leq t^{-2} |\delta' - \delta| C(T) (1 + z^2) (p_{t+\delta}(z) + p_{t+\delta'}(z)). \quad (5.7)$$

Then for $0 < \alpha < 1/4$,

$$\begin{aligned} & \int_{\mathbb{R}} e^{-2|x|} dx \int_{\mathbb{R}} (p_{t+\delta}(x-y) - p_{t+\delta'}(x-y))^{\alpha+2-\alpha} e^{|y|} dy \\ & \leq |\delta' - \delta|^\alpha t^{-2\alpha} C(T, \alpha) \int_{\mathbb{R}} e^{-2|x|} dx \int_{\mathbb{R}} (1 + (x-y)^2)^\alpha (p_{t+\delta}(x-y)^2 + p_{t+\delta'}(x-y)^2) e^{|y|} dy \\ & \leq |\delta' - \delta|^\alpha t^{-2\alpha-1/2} C(T, \alpha). \end{aligned}$$

Consequently,

$$\lim_{\delta, \delta' \rightarrow 0+} \int_0^T e^{-\lambda t} dt \int_{\mathbb{R}} e^{-2|x|} dx \left(\int_{\mathbb{R}} X_0(dy) p_{t+\delta}(x-y) - \int_{\mathbb{R}} X_0(dy) p_{t+\delta'}(x-y) \right)^2 = 0.$$

By Equation (5.7), for $0 < \alpha < 1/4$, we have

$$\begin{aligned} & \int_{\mathbb{R}} X_0(dy) \int_{\mathbb{R}} p_{t-s}(y-z) (p_{s+\delta}(x-z) - p_{s+\delta'}(x-z))^2 dz \\ & \leq (2\pi(t-s))^{-1/2} \int_{\mathbb{R}} X_0(dy) \int_{\mathbb{R}} (p_{s+\delta}(x-z) - p_{s+\delta'}(x-z))^{\alpha+2-\alpha} dz \\ & \leq (2\pi(t-s))^{-1/2} s^{-2\alpha} |\delta - \delta'|^\alpha C(T, \alpha) \int_{\mathbb{R}} (1 + (x-z)^2)^\alpha (p_{s+\delta}(x-z)^2 + p_{s+\delta'}(x-z)^2) dz \\ & \leq |\delta - \delta'|^\alpha C(T, \alpha) (t-s)^{-1/2} s^{-2\alpha-1/2}. \end{aligned}$$

Therefore,

$$\lim_{\delta, \delta' \rightarrow 0+} \int_0^T dt \int_{\mathbb{R}} e^{-2|x|} dx \int_0^t \lambda e^{-\lambda s} ds \int_{\mathbb{R}} X_0(dy) \int_{\mathbb{R}} p_{t-s}(y-z) (p_{s+\delta}(x-z) - p_{s+\delta'}(x-z))^2 dz = 0.$$

Then

$$\lim_{\delta, \delta' \rightarrow 0+} \mathbb{E} \int_0^T \int_{\mathbb{R}} (T_{\delta} X_t(x) - T_{\delta'} X_t(x))^2 e^{-2|x|} dx dt = 0$$

and the desired result follows. \square

Although for a.e. s , the function $h_s(x)$ is defined for a.e. $x \in \mathbb{R}$, the quantity $h_s(W_s^{t,x})$ is a well-defined random variable when $s > t$ since the law of $W_s^{t,x}$ is absolutely continuous.

Theorem 5.4 *Suppose that $\partial_x u_0 \in L^2(\mathbb{R}, e^{-|x|} dx)$. If $\{v_t(x)\}$ is a solution to (5.3) such that $v_t(x)$ is differentiable in x and*

$$\mathbb{E} \int_0^T \int_{\mathbb{R}} (\partial_x v_t(x))^2 e^{-2|x|} dx dt < \infty,$$

then $v_t(x) = Y_t^{t,x}$ a.s., where $\{Y_s^{t,x}\}$ is a solution to the BTSDE (5.4).

Sketch of the proof Since the main idea is the same as that of Theorem 4.1 in [25], we only give a sketch. Applying T_{δ} to both sides of (5.3), we get for $v_s^{\delta} := T_{\delta} v_s$

$$\begin{aligned} v_t^{\delta}(x) &= u_0^{\delta}(x) + \int_t^T \int_{[0,1]} \int_{\mathbb{R}} G_1(y, v_{s+}(l)) p_{\delta}(x-l) dl \tilde{B}(\hat{d}s dy) \\ &\quad + \int_t^T \int_{\Delta} \int_{[0,1]^N} \int_{\mathbb{R}} G_2(y, z, v_{s+}(l)) p_{\delta}(x-l) dl \tilde{M}(\hat{d}s dz dy) + \int_t^T \frac{1}{2} \Delta v_{s+}^{\delta}(x) ds, \end{aligned}$$

where G_1 and G_2 are the obvious integrands of (5.3).

Let $s = t_0 < t_1 < \dots < t_n = T$ be a partition of $[s, T]$. Writing $T_{\delta} v_s(W_s^{t,x}) - T_{\delta} v_T(W_T^{t,x})$ into telescopic sum, we have

$$\begin{aligned} &v_s^{\delta}(W_s^{t,x}) - T_{\delta} u_0(W_T^{t,x}) \\ &= \sum_{i=0}^{n-1} \left(v_{t_i}^{\delta}(W_{t_i}^{t,x}) - v_{t_i}^{\delta}(W_{t_{i+1}}^{t,x}) \right) + \sum_{i=0}^{n-1} \left(v_{t_i}^{\delta}(W_{t_{i+1}}^{t,x}) - v_{t_{i+1}}^{\delta}(W_{t_{i+1}}^{t,x}) \right) \\ &= - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{2} \Delta v_{t_i}^{\delta}(W_r^{t,x}) dr - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \nabla v_{t_i}^{\delta}(W_r^{t,x}) dW_r \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{[0,1]} \int_{\mathbb{R}} p_{\delta}(W_{t_{i+1}}^{t,x} - l) G_1(z, v_{r+}(l)) dl \tilde{B}(\hat{d}r dz) \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{2} \Delta v_{r+}^{\delta}(W_{t_{i+1}}^{t,x}) dr \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{\Delta} \int_{[0,1]^N} \int_{\mathbb{R}} p_{\delta}(W_{t_{i+1}}^{t,x} - l) G_2(y, z, v_{r+}(l)) dl \tilde{M}(\hat{d}r dz dy). \end{aligned}$$

Letting the mesh size decrease to zero and taking $\delta \rightarrow 0$, we see that

$$Y_s^{t,x} = v_s(W_s^{t,x}) \text{ and } Z_s^{t,x} = \partial_x v_s(W_s^{t,x}), \quad \text{a.e. } (s, \omega) \in (t, T] \times \Omega,$$

is a solution to the BTSDE (5.4). By the continuity of $Y_s^{t,x}$ in s and the continuity of $v_s(W_s^{t,x})$ at $s = t$, taking $s \downarrow t$, we get the conclusion of the theorem. \square

Applying Yamada-Watanabe's argument, we can get the following result.

Theorem 5.5 *The BTSDE (5.4) has at most one solution.*

Proof. We drop the superscript (t, x) in (5.4) for simplicity. Suppose that (5.4) has two solutions (Y_t^i, Z_t^i) , $i = 1, 2$. Let (a_k) be a sequence of decreasing positive numbers defined recursively by

$$a_0 = 1 \text{ and } \int_{a_k}^{a_{k-1}} z^{-1} dz = k, \quad k \geq 1.$$

Let ψ_k be non-negative continuous functions supported in (a_k, a_{k-1}) satisfying

$$\int_{a_k}^{a_{k-1}} \psi_k(z) dz = 1 \text{ and } \psi_k(z) \leq 2(kz)^{-1}, \quad \forall z \in \mathbb{R}.$$

Let

$$\phi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x) dx, \quad \forall z \in \mathbb{R}.$$

Then, $\phi_k(z) \rightarrow |z|$ and $|z|\phi_k''(z) \leq 2k^{-1}$.

Denote $U_1 = [0, 1]$ and $U_2 = \Delta \times [0, 1]^{\mathbb{N}}$. Let G_1 and G_2 be the integrands for the backward stochastic integrals in (5.4). Note that

$$\begin{aligned} Y_t^1 - Y_t^2 &= \int_t^T \int_{U_1} (G_1(u, Y_s^1) - G_1(u, Y_s^2)) B(\hat{ds}du) - \int_t^T (Z_s^1 - Z_s^2) dW_s \\ &\quad + \int_t^T \int_{U_2} (G_2(u, Y_s^1) - G_2(u, Y_s^2)) \tilde{M}(\hat{ds}du). \end{aligned}$$

By Itô's formula, we get

$$\begin{aligned} &\phi_k(Y_t^1 - Y_t^2) \\ &= \int_t^T \int_{U_1} \phi_k'(Y_s^1 - Y_s^2) (G_1(u, Y_s^1) - G_1(u, Y_s^2)) B(\hat{ds}du) \\ &\quad - \int_t^T \phi_k'(Y_s^1 - Y_s^2) (Z_s^1 - Z_s^2) dW_s \\ &\quad + \int_t^T \int_{U_2} (\phi_k(Y_{s+}^1 - Y_{s+}^2 + G_2(u, Y_{s+}^1) - G_2(u, Y_{s+}^2)) - \phi_k(Y_{s+}^1 - Y_{s+}^2)) \tilde{M}(\hat{ds}du) \\ &\quad + \frac{1}{2} \int_t^T \int_{U_1} \phi_k''(Y_s^1 - Y_s^2) (G_1(u, Y_s^1) - G_1(u, Y_s^2))^2 \mu_1(du) ds \\ &\quad + \int_t^T \int_{U_2} \left(\phi_k(Y_s^1 - Y_s^2 + G_2(u, Y_s^1) - G_2(u, Y_s^2)) \right. \\ &\quad \quad \left. - \phi_k(Y_s^1 - Y_s^2) - \phi_k'(Y_{s+}^1 - Y_{s+}^2)(G_2(u, Y_s^1) - G_2(u, Y_s^2)) \right) \mu_2(du) ds \\ &\quad - \frac{1}{2} \int_t^T \phi_k''(Y_s^1 - Y_s^2) (Z_s^1 - Z_s^2)^2 ds. \end{aligned} \tag{5.8}$$

Note that

$$\int_{U_i} |G_i(u, y_1) - G_i(u, y_2)|^2 \mu_i(du) \leq K|y_1 - y_2|.$$

Taking expectation on both sides of (5.8), we get

$$\mathbb{E}\phi_k(Y_t^1 - Y_t^2) \leq K\mathbb{E} \int_t^T \phi_k''(Y_s^1 - Y_s^2)|Y_s^1 - Y_s^2|ds \leq KTk^{-1}.$$

Taking $k \rightarrow \infty$ and making use of Fatou's lemma, we have

$$\mathbb{E}|Y_t^1 - Y_t^2| \leq 0.$$

Therefore, $Y_t^1 = Y_t^2$ a.s. The uniqueness for Z follows easily. \square

As a consequence of the last two theorems, we see that the SPDE (5.2) has a unique strong solution.

6 Appendix

We consider the case of $E = \{e_1, e_2\}$ with $\nu_0(e_1) = \nu_0(e_2) = 1/2$ in this Appendix. Let $F = \{e_1\}$.

For convenience, we denote $a_2 = \beta_{2;2;0} + \sigma^2$, $a_i = \beta_{i;i;0}$, $i = 3, 4, 5, 6$, $a_{21} = \beta_{3;2;1} + \sigma^2$, $a_{211} = \beta_{4;2;2} + \sigma^2$, $a_{22} = \beta_{4;2;2;0}$, $a_{31} = \beta_{4;3;1}$, $a_{2111} = \beta_{5;2;3} + \sigma^2$, $a_{221} = \beta_{5;2;2;1}$, $a_{311} = \beta_{5;3;2}$, $a_{32} = \beta_{5;3;2;0}$, $a_{41} = \beta_{5;4;1}$, $a_{21111} = \beta_{6;2;4} + \sigma^2$, $a_{2211} = \beta_{6;2;2;2}$, $a_{3111} = \beta_{6;3;3}$, $a_{222} = \beta_{6;2;2;2;0}$, $a_{321} = \beta_{6;3;2;1}$, $a_{411} = \beta_{6;4;2}$, $a_{33} = \beta_{6;3;3;0}$, $a_{42} = \beta_{6;4;2;0}$, $a_{51} = \beta_{6;5;1}$. Let

$$\mathbf{m}_i = \int_{M_1(E)} \mu^i(\{e_1\})\Pi(d\mu),$$

where $i = 1, 2, \dots, 6$ and Π is the invariant measure. By the consistency condition, we have

$$\left\{ \begin{array}{l} a_{21} = a_2 - a_3, \\ a_{22} = a_2 - 2a_3 + a_4 - a_{211}, \\ a_{31} = a_3 - a_4, \\ a_{41} = a_4 - a_5, \\ a_{221} = \frac{1}{5}a_2 - \frac{4}{5}a_3 + a_4 + \frac{1}{5}a_{211} - \frac{2}{5}a_{2111} - \frac{2}{5}a_5, \\ a_{311} = \frac{3}{5}a_{211} - \frac{2}{5}a_2 + \frac{8}{5}a_3 - 2a_4 - \frac{1}{5}a_{2111} + \frac{4}{5}a_5, \\ a_{32} = -\frac{3}{5}a_3 + \frac{1}{5}a_{2111} + \frac{1}{5}a_5 - \frac{3}{5}a_{211} + \frac{2}{5}a_2, \\ a_{2211} = -\frac{14}{15}a_3 + a_4 + \frac{2}{5}a_2 - \frac{4}{5}a_5 - \frac{2}{3}a_{33} - a_{42} - \frac{3}{5}a_{211} + \frac{8}{15}a_{2111} - \frac{1}{3}a_{21111} + \frac{1}{3}a_6, \\ a_{222} = 2a_{211} - a_2 + \frac{4}{3}a_3 + \frac{8}{3}a_{33} + 3a_{42} - \frac{4}{3}a_{2111} + \frac{1}{3}a_{21111} - \frac{1}{3}a_6, \\ a_{3111} = -\frac{3}{5}a_{2111} + \frac{14}{5}a_3 - 3a_4 - \frac{6}{5}a_2 + \frac{12}{5}a_5 + 2a_{33} + 3a_{42} + \frac{9}{5}a_{211} - a_6, \\ a_{321} = \frac{2}{5}a_2 - \frac{3}{5}a_3 + \frac{1}{5}a_5 - a_{33} - a_{42} - \frac{3}{5}a_{211} + \frac{1}{5}a_{2111}, \\ a_{411} = a_4 - 2a_5 - a_{42} + a_6, \\ a_{51} = a_5 - a_6. \end{array} \right. \quad (6.1)$$

Taking $p = 1$, $q = 0$ and $f = 1_F$ in Equation (4.2), we get

$$\mathbf{m}_1 = \alpha = 1/2. \quad (6.2)$$

Taking $p = 2$, $q = 0$ and $f = 1_{F \times F}$ in Equation (4.2), we get

$$\mathbf{m}_2 = \frac{a_2 + \theta\alpha}{a_2 + \theta} \mathbf{m}_1 = \frac{2a_2 + \theta}{4(a_2 + \theta)}. \quad (6.3)$$

Taking $p = 3$, $q = 0$ and $f = 1_{F \times F \times F}$ in Equation (4.2), we get

$$\mathbf{m}_3 = \frac{a_3 \mathbf{m}_1 + (3a_{21} + \frac{3}{2}\theta\alpha)\mathbf{m}_2}{3a_{21} + a_3 + \frac{3}{2}\theta} = \frac{4a_2 + \theta}{8(a_2 + \theta)}, \quad (6.4)$$

where we have applied the consistency condition $a_{21} = a_2 - a_3$ and canceled the common positive factor

$$3a_2 - 2a_3 + \frac{3}{2}\theta = 3a_{21} + a_3 + \frac{3}{2}\theta$$

from both the denominator and the numerator.

Taking $p = 4$, $q = 0$ and $f = 1_{F \times F \times F \times F}$ in Equation (4.2), we get

$$\begin{aligned} \mathbf{m}_4 &= \frac{a_4 \mathbf{m}_1 + (4a_{31} + 3a_{22})\mathbf{m}_2 + (6a_{211} + 2\theta\alpha)\mathbf{m}_3}{6a_{211} + 3a_{22} + 4a_{31} + a_4 + 2\theta} \\ &= \frac{\theta^2 - 4a_3\theta + 10a_2\theta + 2a_4\theta + 12a_{211}a_2 + 12a_2^2 - 8a_3a_2}{8(3a_{211} + 3a_2 - 2a_3 + 2\theta)(a_2 + \theta)} \\ &= \frac{\theta^2 - 4a_3\theta + 10a_2\theta + 2a_4\theta + 12a_{211}a_2 + 12a_2^2 - 8a_3a_2}{8(6a_{211} + 3a_{22} + 4a_{31} + a_4 + 2\theta)(a_2 + \theta)}, \end{aligned} \quad (6.5)$$

where $3a_{211} + 3a_2 - 2a_3 + 2\theta = 6a_{211} + 3a_{22} + 4a_{31} + a_4 + 2\theta$ follows from the consistency condition.

Taking $p = 1$, $q = 3$, $f = 1_F$ and $g = 1_{F \times F \times F}$ in Equation (4.2), we get

$$(3a_{21} + \theta\alpha)\mathbf{m}_3 + a_3\mathbf{m}_2 - (3a_{21} + a_3 + \theta)\mathbf{m}_4 = 0. \quad (6.6)$$

Plugging in Equations (6.3)-(6.5), Equation (6.6) becomes

$$\frac{\theta B}{(3a_{211} + 3a_2 - 2a_3 + 2\theta)(a_2 + \theta)} = \frac{\theta B}{(6a_{211} + 3a_{22} + 4a_{31} + a_4 + 2\theta)(a_2 + \theta)} = 0,$$

where

$$B := (-6a_3 + 3a_2 - 3a_{211} + 4a_4)\theta + 12a_4a_2 - 22a_3a_2 + 6a_3a_{211} - 6a_{211}a_2 - 8a_4a_3 + 12a_3^2 + 6a_2^2.$$

By the consistency condition, we have

$$-6a_3 + 3a_2 - 3a_{211} + 4a_4 = a_4 + 3a_{22}.$$

If $a_4 + 3a_{22} = 0$, then the Ξ -coalescent degenerates to Kingman's coalescent by Lemma 4.3; otherwise, $a_4 + 3a_{22} > 0$ and we have

$$\theta = \frac{2(6a_4a_2 - 11a_3a_2 + 3a_3a_{211} - 3a_{211}a_2 - 4a_4a_3 + 6a_3^2 + 3a_2^2)}{-a_4 - 3a_{22}}, \quad (6.7)$$

which further imposes a necessary condition for the (Ξ, A) -Fleming-Viot process in case of $E = \{e_1, e_2\}$ with $\nu_0(\{e_1\}) = \nu_0(\{e_2\}) = 1/2$.

With Equation (6.7) we can compute higher moments to reach a contradiction. Taking $p = 5$, $q = 0$ and $f = 1_{F \times F \times F \times F \times F}$ in Equation (4.2), we get

$$\begin{aligned} \mathbf{m}_5 &= \frac{a_5 \mathbf{m}_1 + (10 a_{32} + 5 a_{41}) \mathbf{m}_2 + (15 a_{221} + 10 a_{311}) \mathbf{m}_3 + (10 a_{2111} + \frac{5}{2} \theta \alpha) \mathbf{m}_4}{10 a_{2111} + 15 a_{221} + 10 a_{311} + 10 a_{32} + 5 a_{41} + a_5 + \frac{5}{2} \theta} \\ &= \frac{(\theta - 6 a_{211} + 10 a_4 + 20 a_2 - 16 a_3) \theta - 16 a_3 a_2 + 24 a_2^2 + 24 a_{211} a_2}{16 (a_2 + \theta) (3 a_{211} + 3 a_2 - 2 a_3 + 2 \theta)} \\ &= \frac{(\theta - 6 a_{211} + 10 a_4 + 20 a_2 - 16 a_3) \theta - 16 a_3 a_2 + 24 a_2^2 + 24 a_{211} a_2}{16 (a_2 + \theta) (6 a_{211} + 3 a_{22} + 4 a_{31} + a_4 + 2 \theta)}, \end{aligned} \quad (6.8)$$

where we have applied the consistency condition and canceled a common positive factor

$$4 a_{2111} + 3 a_2 - 2 a_3 + 3 a_{211} + \frac{5}{2} \theta = 10 a_{2111} + 15 a_{221} + 10 a_{311} + 10 a_{32} + 5 a_{41} + a_5 + \frac{5}{2} \theta$$

from both the denominator and numerator.

Taking $p = 6$, $q = 0$ and $f = 1_{F \times F \times F \times F \times F \times F}$ in Equation (4.2), we get

$$\begin{aligned} \mathbf{m}_6 &= \frac{a_6 \mathbf{m}_1 + (6 a_{51} + 15 a_{42} + 10 a_{33}) \mathbf{m}_2 + (15 a_{411} + 60 a_{321} + 15 a_{222}) \mathbf{m}_3}{C} \\ &\quad + \frac{(20 a_{3111} + 45 a_{2211}) \mathbf{m}_4 + (15 a_{21111} + 3 \theta \alpha) \mathbf{m}_5}{C}, \end{aligned} \quad (6.9)$$

where

$$C := a_6 + 6 a_{51} + 15 a_{42} + 10 a_{33} + 15 a_{411} + 60 a_{321} + 15 a_{222} + 20 a_{3111} + 45 a_{2211} + 15 a_{21111} + 3 \theta.$$

Taking $p = 1$, $q = 5$, $f = 1_F$ and $g = 1_{F \times F \times F \times F \times F}$ in Equation (4.2), we get

$$\begin{aligned} &a_5 \mathbf{m}_2 + (10 a_{32} + 5 a_{41}) \mathbf{m}_3 + (10 a_{311} + 15 a_{221}) \mathbf{m}_4 + (10 a_{2111} + 2 \theta \alpha) \mathbf{m}_5 \\ &- (a_5 + 10 a_{32} + 5 a_{41} + 10 a_{311} + 15 a_{221} + 10 a_{2111} + 2 \theta) \mathbf{m}_6 = 0. \end{aligned} \quad (6.10)$$

Plugging in Equations (6.3), (6.4), (6.5), (6.8), (6.9) and (6.7), Equation (6.10) only involves the collision rates. We can then consider examples of coalescents in which those rates are specified.

Proof of Proposition 4.5 Let $\mathbb{M}_n = \int_0^1 x^n \Xi(dx)$. It follows that

$$\mathbb{M}_n = \frac{\Gamma(n+2-\beta)}{(n+1)! \Gamma(2-\beta)}.$$

By integral representation of the coalescence rates, we have

$$\begin{aligned} a_2 &= \mathbb{M}_0, a_3 = \mathbb{M}_1, a_4 = \mathbb{M}_2, a_{211} = \mathbb{M}_0 - 2\mathbb{M}_1 + \mathbb{M}_2, \\ a_5 &= \mathbb{M}_3, a_{2111} = \mathbb{M}_0 - 3\mathbb{M}_1 + 3\mathbb{M}_2 - \mathbb{M}_3, a_{42} = 0, \\ a_{33} &= 0, a_6 = \mathbb{M}_4, a_{21111} = \mathbb{M}_0 - 4\mathbb{M}_1 + 6\mathbb{M}_2 - 4\mathbb{M}_3 + \mathbb{M}_4. \end{aligned}$$

The value of θ can be obtained by Equation (6.7). Plugging in the values of coalescence rates and θ to Equation (6.10), we have

$$\frac{(\beta-2)(\beta-3)(\beta+1)(\beta^4+8\beta^3-39\beta^2+6\beta+72)\beta^2}{(\beta^2+3)(\beta^4+6\beta^3-\beta^2-126\beta-72)} = 0.$$

We can not find any $\beta \in (0, 2)$ to satisfy the above equation, which contradicts the reversibility. Therefore, the $(\text{Beta}(2-\beta, \beta), A)$ -Fleming-Viot process is not reversible. \square

Proof of Proposition 4.6 Let $\mathbb{M}_n = \int_0^1 x^n \Xi(dx)$. We have

$$\mathbb{M}_n = 1/(n+1-\gamma).$$

The values of $a_2, a_3, a_4, a_{211}, a_5, a_{2111}, a_6, a_{21111}$ and θ can be similarly expressed in terms of γ . Note that $a_{42} = a_{33} = 0$. With the consistency condition (6.1) and the moments obtained from Equations (6.2)-(6.5), (6.8)-(6.9), we plug in those values to Equation (6.10). It follows that

$$\frac{-1+\gamma}{(-4+\gamma)(-6+\gamma)(\gamma^3-14\gamma^2+61\gamma-120)} = 0.$$

There is no $\gamma \in (0,1)$ satisfying the above equation. Therefore, such a (Ξ, A) -Fleming-Viot process is not reversible. \square

Proof of Proposition 4.7 If $\Xi = \delta_1$, the corresponding coalescent only allows all the blocks to merge into a single block. Then we have $a_2 = 1, a_3 = 1, a_{21} = 0, a_4 = 1, a_{31} = 0, a_{22} = 0$ and $a_{211} = 0$. By Equations (6.3)-(6.5), the moments $\mathbf{m}_2, \mathbf{m}_3$ and \mathbf{m}_4 can be expressed by θ . Plugging in these values to Equation (6.6), we have

$$\frac{\theta^2}{(1+\theta)(1+2\theta)} = 0.$$

Since θ is positive, we get a contradiction. Therefore, the corresponding (Ξ, A) -Fleming-Viot process is not reversible. \square

Proof of Proposition 4.8 By Equation (4.9), the coalescence rates $a_2, a_3, a_{21}, a_4, a_{31}, a_{22}, a_{211}, a_5, a_{41}, a_{32}, a_{311}, a_{221}, a_{2111}, a_6, a_{51}, a_{42}, a_{33}, a_{411}, a_{321}, a_{222}, a_{3111}, a_{2211}$ and a_{21111} are all available. Then the moments $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5$ and \mathbf{m}_6 can be expressed by ϵ and θ . The value of θ can be obtained by Equation (6.7). Replacing all of these values in Equation (6.10), it follows that

$$\frac{(10\epsilon^2 + 11\epsilon + 6)\epsilon^2}{(5\epsilon + 6)(6 + 11\epsilon)(17\epsilon^4 + 109\epsilon^3 + 319\epsilon^2 + 394\epsilon + 120)(1 + \epsilon)} = 0.$$

We can not find any positive ϵ to satisfy the above equation. Therefore, the corresponding (Ξ, A) -Fleming-Viot process is not reversible. \square

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